Bose condensation for the Wu-Austin Hamiltonian without pumping

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(Received 24 February 1992)

The eigenstates of the Wu-Austin Hamiltonian without coupling to the pumping system are considered. We are able to show that for a high quantum number N the energy of the system is proportional to $-N^2$. This has dramatic effects for the entire system at any temperature. We have Bose condensation even without pumping. The rate equations defined by Fröhlich do not describe this behavior.

PACS number(s): 87.10.+e, 87.15.−v, 87.22.−q, 87.50.Jk

I. INTRODUCTION

Fröhlich [1–8] developed a theory of long-range coherence of longitudinal-polarization modes to explain certain anomalous dielectric properties in biological cells. Biological tissues are considered to be open systems with possible stationary states far from equilibrium. Fröhlich considered a model consisting of three parts: (a) A system of oscillators (Bose particles) coupled linearly and nonlinearly to (b) the so-called heat bath. It consists of oscillators and has a certain temperature T. (c) A third system which can pump quanta into system a incoherently. Its effect is described by a so-called pumping rate. His model therefore describes an open system. Because of the pumping, there must be an energy flux from system a into the heat bath.

Fröhlich *postulated* the following rate equation to describe this model:

$$\dot{n}_i = S_i - \Phi_i[n_i \exp(\hbar\omega_i/kT) - (1+n_i)] - \sum_j \chi_{ij}[n_i(1+n_j)\exp(\hbar\omega_i/kT) - n_j(1+n_i)\exp(\hbar\omega_j/kT)] \ . \quad (1.1)$$

Solving this rate equation, Fröhlich could show the following for the stationary state:

(a) If the pumping rate $S=\sum_i S_i$ is zero, which corresponds to the situation where there is no coupling to the third system, the whole system will be in thermal equilibrium. Then the oscillators of system a show the well-known thermal distributions of *free* Bose particles:

$$n_i^T = \frac{1}{e^{\beta\omega_i}-1} \ .$$

(b) If the pumping rate S exceeds a certain value, the oscillators of system a should show the so-called Bose condensation. Almost all quanta are found in the oscillator of the lowest frequency ω_0:

$$n_i \sim \delta_{i,0} \ .$$

Fröhlich explains that then in this new phase *coherent oscillations* of this oscillator could appear.

Wu and Austin [9,10] presented a Hamiltonian from which they derived a rate equation of the form (1.1). Mills [11] derived an extended form of the rate equation using a more general Hamiltonian. Tuszyński, Paul, Chatterjee, and Sreenivasan [12] used the Wu-Austin Hamiltonian to show a formal equivalence to the so-called Davydov Hamiltonian, which was invented by Davydov [13] to explain energy transport in proteins. Recently there was disagreement between Mills [14] on one side and Tuszyński and Paul [15] on the other regarding the question of whether it really is possible to compare Fröhlich's theory, which describes an open system, with Davydov's theory, which describes a closed system and contains no pumping.

In this paper we raise the question of whether the Wu-Austin Hamiltonian can really describe Fröhlich's system. Wu and Austin used perturbation theory to derive the rate equations, but in our opinion perturbation theory should be used carefully if applied to a nonlinear system. Our method used here is independent of perturbation theory. We want to investigate the thermodynamics of the Wu-Austin system, which means that we consider the system without pumping. Our results can then be compared with the corresponding solution of the rate equation. We get totally different results in both cases. We will show that even without pumping, the Wu-Austin system shows Bose condensation and has an energy gap.

The Wu-Austin Hamiltonian [9,10] is defined by

$$H = H_a + H_b + H_p + H_{ab} + H_{aab} + H_{ap} \ , \quad (1.2)$$

with

$$H_a = \sum_{i=1}^{M} \hbar\omega_i a_i^\dagger a_i \ , \quad (1.3)$$

$$H_b = \sum_k \hbar\Omega_k b_k^\dagger b_k \ , \quad (1.4)$$

$$H_p = \sum_j \hbar\theta_j p_j^\dagger p_j \ , \quad (1.5)$$

$$H_{ab} = \hbar \sum_{i=1}^{M} \sum_k (\lambda b_k^\dagger a_i + \lambda^* b_k a_i^\dagger) \ , \quad (1.6)$$

$$H_{aab} = \tfrac{1}{2}\hbar \sum_{i,j=1}^{M} \sum_k (\chi a_j^\dagger a_i b_k^\dagger + \chi^* a_i^\dagger a_j b_k) \ , \quad (1.7)$$

$$H_{ap} = \hbar \sum_{i=1}^{M} \sum_j (\xi p_j a_i^\dagger + \xi^* p_j^\dagger a_i) \ . \quad (1.8)$$

Different temperatures of the subsystems should be necessary for the Bose condensation described by Fröhlich. Thus the heat bath H_b should be at the temperature T_b and the pumping system H_p at the temperature T_p, with $T_p > T_b$. We would then have an energy flux from p to b through a. The correct rate equations should then describe the time dependence of the quantity $n_i = \langle a_i^\dagger a_i \rangle$ with a stationary (asymptotic) solution n_i^0. In this paper we consider the special case of thermal equilibrium ($T = T_p = T_b$); we will determine the distribution n_i^T for this special case and compare it with n_i^0. We will simplify our calculation by neglecting the interaction with the pumping system p. Thus we set $\xi = 0$ in this paper.

In the following we assume that the number of the oscillators a_i should be finite ($M < \infty$). On the other hand, the number of heat-bath modes must be infinite. Indeed, a realistic heat bath must have a continuous frequency spectrum. Therefore, in this paper a sum over k should be read as an integral over k ($\sum_k \cdots = \int d^3k \cdots$). A usual assumption [16] for the frequency distribution is

$$\Omega_k \to \text{const} \times k \quad \text{as } k \to 0 \tag{1.9}$$

for low frequencies and a cutoff for a high frequencies,

$$\Omega_k < \infty \ . \tag{1.10}$$

II. AN EXACT TREATMENT

Before we consider the general case, we use a simplified system in the following: (i) We set $\lambda = 0$, which means we neglect H_{ab}. (ii) We neglect the i dependence of ω_i, so we can write $\omega_i = \omega$. The simplified Hamiltonian (1.2) then reads

$$H = \sum_{i=1}^{M} \hbar\omega a_i^\dagger a_i + \sum_k \hbar\Omega_k b_k^\dagger b_k + \tfrac{1}{2}\hbar \sum_{i,j=1}^{M} \sum_k (\chi a_j^\dagger a_i b_k^\dagger + \chi^* a_i^\dagger a_j b_k) \ . \tag{2.1}$$

Introducing new operators A_l defined by

$$A_l = \sum_{i=1}^{M} O_{l,i} a_i, \quad l = 0, 1, \ldots, M-1 \ , \tag{2.2}$$

with

$$O^T = \begin{pmatrix} \frac{1}{\sqrt{M}} & \frac{1}{\sqrt{M}} & \cdots & \frac{1}{\sqrt{M}} & \frac{1}{\sqrt{M}} \\ -\left[\frac{M-1}{M}\right]^{1/2} & \frac{1}{\sqrt{(M-1)M}} & \cdots & \frac{1}{\sqrt{(M-1)M}} & \frac{1}{\sqrt{(M-1)M}} \\ 0 & -\left[\frac{M-2}{M-1}\right]^{1/2} & \cdots & \frac{1}{\sqrt{(M-2)(M-1)}} & \frac{1}{\sqrt{(M-2)(M-1)}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\sqrt{\tfrac{1}{2}} & \sqrt{\tfrac{1}{2}} \end{pmatrix} , \tag{2.3}$$

we get the following form:

$$\begin{aligned} H &= H_{A_l} + H_b + H_{bA_0} \ , \\ H_{A_l} &= \sum_{l=0}^{M-1} \hbar\omega A_l^\dagger A_l \ , \\ H_b &= \sum_k \hbar\Omega_k b_k^\dagger b_k \ , \\ H_{bA_0} &= \tfrac{1}{2}\hbar M A_0^\dagger A_0 \sum_k (\chi b_k^\dagger + \chi^* b_k) \ . \end{aligned} \tag{2.4}$$

Only the *collective* oscillator ($A_0^\dagger A_0$) feels the interaction with system b.

The eigenstates $|\Psi\rangle$,

$$H|\Psi\rangle = E|\Psi\rangle \ , \tag{2.5}$$

of (2.4) can be calculated exactly. Let us first try the form

$$|\Psi(N)\rangle = \sum_{l=0}^{M-1} (A_l^\dagger)^{N_l} |0\rangle |\beta\rangle \ . \tag{2.6}$$

Using the eigenvalue equation (2.5) we get

$$\begin{aligned} H|\Psi(N)\rangle &= \left[\hbar\omega \sum_{n=0}^{M-1} N_l + \sum_k \hbar\Omega_k b_k^\dagger b_k + \tfrac{1}{2}\hbar M N_0 \sum_k (\chi b_k^\dagger + \chi^\dagger b_k)\right] |\Psi(N)\rangle \\ &= \left[\hbar\omega \sum_{n=0}^{M-1} N_l + \sum_k \hbar\Omega_k [(b_k^\dagger - \beta_k^*)(b_k - \beta_k) - \beta_k^* \beta_k]\right] |\Psi(N)\rangle \ , \end{aligned} \tag{2.7}$$

with

$$\beta_k = -\frac{MN_0\chi}{2\Omega_k} \ . \tag{2.8}$$

This gives an eigenvalue equation for $|\beta\rangle$:

$$\left[\sum_k \hbar\Omega_k[(b_k^\dagger-\beta_k^*)(b_k-\beta_k)-\beta_k^*\beta_k]\right]|\beta\rangle = E_b|\beta\rangle \ . \tag{2.9}$$

If we are interested in the lowest state $|\Psi(N)\rangle$ (for fixed N), $|\beta\rangle$ must be the so-called quasiclassical state:

$$b_k|\beta\rangle = \beta_k|\beta\rangle \ . \tag{2.10}$$

The eigenvalue $E(N)$ is

$$E(N) = \sum_{l=0}^{M-1} \hbar\omega N_l - \hbar N_0^2 M^2|\chi|^2 \sum_k \frac{1}{4\Omega_k} \ . \tag{2.11}$$

The total system of eigenstates $|\Psi(N,n)\rangle$ of H is given by a generalization of (2.6):

$$|\Psi(N,n)\rangle = \prod_{l=0}^{M-1}(A_l^\dagger)^{N_l}|0\rangle \prod_k (B_k^\dagger)^{n_k}|\beta\rangle \ , \tag{2.12}$$

with

$$B_k|\beta\rangle = 0, \quad B_k = b_k - \beta_k \ . \tag{2.13}$$

The state (2.12) has the energy $E(N,n)$:

$$\begin{aligned} E(N,n) &= \sum_{l=0}^{M-1}\hbar\omega N_l + \sum_k \hbar\Omega_k n_k - \hbar N_0^2 K \ , \\ K &= M^2|\chi|^2 \sum_k \frac{1}{4\Omega_k} \ . \end{aligned} \tag{2.14}$$

The same result can be obtained in a more elegant way if we use a unitary transformation for the Hamiltonian (2.4)

$$\begin{aligned} \bar{H} &= UHU^\dagger \ , \\ U &= \exp\left[MA_0^\dagger A_0 \sum_k \frac{1}{2\Omega_k}(\chi b_k^\dagger - \chi^* b_k)\right] \ . \end{aligned} \tag{2.15}$$

Since

$$\begin{aligned} Ub_kU^\dagger &= b_k - \frac{M\chi}{2\Omega_k}A_0^\dagger A_0 \ , \\ UA_0^\dagger A_0 U^\dagger &= A_0^\dagger A_0 \ , \end{aligned} \tag{2.16}$$

we obtain a very simple form for the transformed Hamiltonian:

$$\bar{H} = \sum_{l=0}^{M-1}\hbar\omega A_l^\dagger A_l + \sum_k \hbar\Omega_k b_k^\dagger b_k - \hbar K(A_0^\dagger A_0)^2 \ . \tag{2.17}$$

The use of the transformations (2.13) or (2.16) can cause difficulties if

$$\int d^3k\,|\beta_k|^2 = \infty \tag{2.18}$$

because the two Hilbert spaces would then be unitarily inequivalent to each other [17]. But from (1.9) and (1.10) it follows that this integral exists in our case. We therefore get the astonishing result that the system can lower its energy without any limitation by increasing N_0. In Sec. IV we will see how dramatic the consequences are.

III. THE GENERAL CASE

Since the general case $H = H_a + H_b + H_{ab} + H_{aab}$ cannot be calculated exactly, we use the following product ansatz:

$$|\Psi\rangle = |\varphi_a\rangle|\varphi_b\rangle \ . \tag{3.1}$$

Using the well-known variational principle to optimize (3.1), we get

$$H_a^{\text{eff}}|\varphi_a\rangle = E_a|\varphi_a\rangle, \quad H_b^{\text{eff}}|\varphi_b\rangle = E_b|\varphi_b\rangle \ , \tag{3.2}$$

with

$$\begin{aligned} H_a^{\text{eff}} = {} & \sum_i \hbar\omega_i a_i^\dagger a_i + \hbar\sum_{i,k}(\lambda\langle b_k^\dagger\rangle a_i + \lambda^*\langle b_k\rangle a_i^\dagger) \\ & + \tfrac{1}{2}\hbar\sum_{i,j} a_i^\dagger a_j \sum_k(\chi\langle b_k^\dagger\rangle + \chi^*\langle b_k\rangle) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} H_b^{\text{eff}} = {} & \sum_i \hbar\Omega_k b_k^\dagger b_k + \hbar\sum_{i,k}(\lambda b_k^\dagger\langle a_i\rangle + \lambda^* b_k\langle a_i^\dagger\rangle) \\ & + \tfrac{1}{2}\hbar\sum_{i,j}\langle a_i^\dagger a_j\rangle\sum_k(\chi b_k^\dagger + \chi^* b_k) \ . \end{aligned} \tag{3.4}$$

H_b^{eff} can be diagonalized into

$$H_b^{\text{eff}} = \hbar\sum_k \Omega_k[(b_k^\dagger - \beta_k^*)(b_k - \beta_k) - \beta_k^*\beta_k] \ , \tag{3.5}$$

with

$$\beta_k = \langle b_k\rangle = -\frac{1}{\Omega_k}\left[\lambda\sum_i\langle a_i\rangle + \tfrac{1}{2}\chi\sum_{i,j}\langle a_i^\dagger a_j\rangle\right] \ . \tag{3.6}$$

H_a^{eff} can be transformed to

$$H_a^{\text{eff}} = \hbar\sum_{i,j}(\omega_i\delta_{ij} - \alpha)[(a_i^\dagger - \alpha_i^*)(a_j - \alpha_j) - \alpha_i^*\alpha_j] \ , \tag{3.7}$$

with

$$\begin{aligned} \alpha_i &= \langle a_i\rangle = \frac{1}{\omega_i}\,\frac{\lambda^*\sum_k\beta_k}{\alpha\sum_i\frac{1}{\omega_i} - 1} \ , \\ \alpha &= -\tfrac{1}{2}\sum_k(\chi\beta_k^* + \chi^*\beta_k) \ . \end{aligned} \tag{3.8}$$

In order to diagonalize (3.7) we need to solve the eigenvalue equation

$$\sum_j(\omega_i\delta_{ij} - \alpha)e_i^l = \lambda_l e_i^l \ . \tag{3.9}$$

For the normalized eigenvectors e_i^l we get

$$e_i^l = \frac{A}{\omega_i - \lambda_l}, \quad A^{-1} = \left[\sum_i \frac{1}{(\omega_i - \lambda_l)^2}\right]^{1/2} \ . \tag{3.10}$$

The eigenvalues are determined by the following equation:

$$\frac{1}{\alpha}=\sum_i \frac{1}{\omega_i-\lambda_l} \ . \tag{3.11}$$

Using the new operators A_l,

$$A_l=\sum_{i=1}^{M} e_i^l(a_i-\alpha_i), \quad l=0,1,\ldots,M-1 \ ,$$
$$a_i=\sum_{l=0}^{M-1} e_i^l A_l+\alpha_i \ , \tag{3.12}$$

we arrive at the simple form

$$H_a^{\text{eff}}=\hbar\sum_{l=0}^{M-1}\lambda_l A_l^\dagger A_l-\hbar\sum_{i,j}(\omega_i\delta_{ij}-\alpha)\alpha_i^*\alpha_j \ . \tag{3.13}$$

Equation (3.11) cannot be solved exactly. We will show later that we always have $\alpha>0$. For this case, we present the graphical solution in Fig. 1. We see that the eigenvalues λ_i for $i\neq 0$ fulfill the following conditions:

$$\omega_i<\lambda_i<\omega_{i+1}, \quad i\neq 0 \ , \tag{3.14}$$

and are therefore bounded. On the other hand, λ_0 is sensitive to the value of α. We will see later that $N_0=\langle A_0^\dagger A_0\rangle\rightarrow\infty$ implies that $\alpha\rightarrow\infty$. For this limiting case we have

$$\lambda_0\rightarrow-\alpha M, \quad e_i^0\rightarrow\frac{1}{\sqrt{M}}, \quad A_0\rightarrow\frac{1}{\sqrt{M}}\sum_i(a_i-\alpha_i) \ . \tag{3.15}$$

From (3.13) it follows that $|\varphi_a\rangle$ can be written as

$$|\varphi_a\rangle=\prod_{l=0}^{M-1}(A_l^\dagger)^{N_l}|0\rangle \ , \tag{3.16}$$

whereas $|\varphi_b\rangle$ has the form

$$|\varphi_b\rangle=\prod_k(b_k^\dagger-\beta_k^*)^{n_k}|0\rangle \ . \tag{3.17}$$

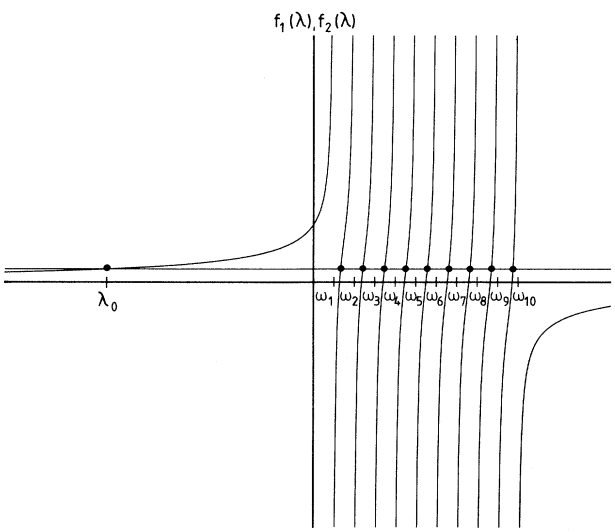

FIG. 1. For the graphical solution of (3.11), we plot the two functions $f_1(\lambda)=\text{const}=1/\alpha$ and $f_2(\lambda)=\sum_i[1/(\omega_i-\lambda)]$ as an exemplary illustration. The marked points of intersection $f_1(\lambda)=f_2(\lambda)$ are the solutions λ_l of (3.11).

The state $|\Psi\rangle=|\varphi_a\rangle|\varphi_b\rangle$ is only dependent on the two quantum numbers

$$N_l=\langle A_l^\dagger A_l\rangle$$
$$n_k=\langle(b_k^\dagger-\beta_k^*)(b_k-\beta_k)\rangle=\langle b_k^\dagger b_k\rangle-\beta_k^*\beta_k \ . \tag{3.18}$$

The remaining quantities β_k, α_i, λ_l, and α have to be determined by (3.6) together with (3.8), (3.10), and (3.11).

The next step is the evaluation of $\langle H\rangle$. From (3.3), (3.13), and (3.16)–(3.18) we derive

$$\langle H\rangle=E(N,n)=\langle H_a^{\text{eff}}\rangle+\hbar\sum_k\Omega_k\langle b_k^*b_k\rangle$$
$$=\hbar\sum_{l=0}^{M-1}\lambda_l N_l-\hbar\sum_{i,j}(\omega_i\delta_{ij}-\alpha)\alpha_i^*\alpha_j$$
$$+\hbar\sum_k\Omega_k(n_k+\beta_k^*\beta_k) \ . \tag{3.19}$$

The calculation of λ_l, α_i, and β_k in general cannot be carried out exactly. However, we can calculate $\langle H\rangle$ in the limiting case $N_0\rightarrow\infty$ to show that the form of $E(N,n)$ is again similar to (2.14):

$$E(N,n)\sim -N_0^2+O(N_0) \ . \tag{3.20}$$

What we have to do is to solve the following set of equations:

$$\beta_k=-\frac{1}{\Omega_k}\left[\lambda\sum_i\alpha_i+\tfrac{1}{2}\chi\sum_{i,j}\langle a_i^\dagger a_j\rangle\right] \ , \tag{3.6$'$}$$

$$\alpha_i=-\frac{1}{\omega_i}\frac{\lambda^*\sum_k\beta_k}{\frac{1}{2}\sum_k(\chi\beta_k^*+\chi^*\beta_k)\sum_i\frac{1}{\omega_i}+1} \ , \tag{3.8$'$}$$

$$\alpha=-\tfrac{1}{2}\sum_k(\chi\beta_k^*+\chi^*\beta_k) \ ,$$

Eqs. (3.10) and (3.11). To solve the first equation we need

$$\sum_{i,j}\langle a_i^\dagger a_j\rangle=\sum_{i,j}\left\langle\left[\sum_l e_i^l A_l^\dagger+\alpha_l^*\right]\right.$$
$$\left.\times\left[\sum_{l'}e_i^{l'}A_{l'}+\alpha_{l'}\right]\right\rangle$$
$$=\sum_l\left[\sum_i e_i^l\right]^2 N_l+\left|\sum_i\alpha_i\right|^2 \ . \tag{3.21}$$

This quantity is positive and goes to $+\infty$ for $N_0\rightarrow\infty$. For this case it follows from (3.6′) and (3.8′) that β_k and α are going to infinity, whereas α_i remains finite. Therefore, the conditions to obtain (3.15) are fulfilled. Thus for $N_0\rightarrow\infty$ the important quantities have the following structure:

$$\lambda_0 = -\alpha M + O(1),$$
$$e_i^0 = \frac{1}{\sqrt{M}} + O(1/N_0),$$
$$\alpha = \tfrac{1}{2}|\chi|^2 N_0 M \sum_k \frac{1}{\Omega_k} + O(1), \tag{3.22}$$
$$\beta_k = -\frac{1}{2\Omega_k}\chi N_0 M + O(1),$$
$$\alpha_i = -\frac{\lambda^*}{\omega_i \chi^* \sum_l 1/\omega_l} + O(1/N_0).$$

In leading terms of N_0 we therefore have

$$\langle H \rangle = -\tfrac{1}{4}\hbar|\chi|^2 N_0^2 M^2 \sum_k \frac{1}{\Omega_k} + O(N_0), \tag{3.23}$$

which is exactly the same dependence as in (2.14). As in Sec. II we see that the Wu-Austin system has the ground-state energy $-\infty$.

IV. THE SYSTEM IN CONTACT WITH A HEAT BATH

Due to the interaction with the heat bath, the occupation numbers N_l and n_k will be found according to the temperature T of the heat bath. Let us first calculate the partition function Z for the system described in Sec. II:

$$Z = \left[\prod_{l=1}^{M-1} Z_l\right]\left[\prod_k Z_k\right] Z_0, \tag{4.1}$$

with

$$Z_l = \sum_{N_l=0}^{\infty} \exp(-\beta\hbar\omega N_l), \quad l=1,\ldots,M-1,$$
$$Z_k = \sum_{n_k=0}^{\infty} \exp(-\beta\hbar\Omega_k n_k), \tag{4.2}$$
$$Z_0 = \sum_{N_0=0}^{\infty} \exp[-\beta\hbar(N_0\omega - N_0^2 K)],$$
$$\beta = \frac{1}{k_B T}.$$

Z_l and Z_k are the well-known partition functions for harmonic oscillators:

$$Z_l = \frac{1}{1-\exp(-\beta\hbar\omega)},$$
$$Z_k = \frac{1}{1-\exp(-\beta\hbar\Omega_k)}. \tag{4.3}$$

Z_0 is infinite. In order to see this in detail we define

$$Z_{0,\bar{N}_0} = \sum_{N_0=0}^{\bar{N}_0} \exp[-\beta\hbar(N_0\omega - N_0^2 K)]. \tag{4.4}$$

For $\bar{N}_0 \to \infty$ we have

$$Z_{0,\bar{N}_0} = e^{-\beta\hbar(\omega\bar{N}_0 - K\bar{N}_0^2)} \times [1 + e^{2\beta\hbar K\bar{N}_0} + (e^{2\beta\hbar K\bar{N}_0})^2 + \cdots]. \tag{4.5}$$

We see that only the last term in (4.4) is important. Since we have to take the limit $\bar{N}_0 \to \infty$, $\langle N_0 \rangle$ is infinite, too, independent of the temperature. The A_0 oscillator shows Bose condensation for all temperatures. The reason for this is an infinite energy gap Δ,

$$\Delta = 2\hbar K\bar{N}_0 = \tfrac{1}{2}\hbar|\chi|^2\bar{N}_0 M^2 \sum_k \frac{1}{\Omega_k} \to \infty, \tag{4.6}$$

departing the ground state ($N_0 = \infty$) from the first excited state.

It is immediately clear that (4.6) holds also in the general case of Sec. III. The nonlinear part H_{aab} of the Hamiltonian dominates the behavior of the system for $N_0 \to \infty$, as we can see from (3.23).

V. DISCUSSION

In this paper we calculate the energy of the Wu-Austin Hamiltonian without pumping. The simplified system ($\lambda=0$, $\omega_i=\omega$) can be computed exactly (Sec. II). The system in general is calculated using the product ansatz (3.1), which turns out to be exact in the simplified model, and considering only the important limiting case of high quantum numbers (Sec. III). We do not use any perturbation theoretical methods. In both cases we obtain the result that the energy of the ground state is proportional to $-N_0^2$. This *negative excitation energy* has the consequence that the system can lower its energy by just increasing the number of quanta N_0. In Sec. IV we show that the system therefore has an infinite energy gap, which means it always remains in the ground state independent of the temperature.

Thus the Wu-Austin system shows Bose condensation at all temperatures even without pumping. This phenomenon can be described as follows: almost all quanta are found in one mode. But this mode is a collective mode and not just the mode with the lowest frequency, as can be seen from Eqs. (2.2) and (2.3) or (3.12) and (3.15). However, the rate equation which Fröhlich postulated (1.1) shows, in the case of vanishing pumping rate, the thermal distribution of *free* Bose particles. Hence this rate equation cannot be derived from the Wu-Austin Hamiltonian.

One can ask why we obtain such different results from Wu and Austin [9,10] and Hirsch [18], who derived the rate equation from the Hamiltonian (1.2), using well known procedures to do so. In our opinion the validity of all these procedures depends on two crucial conditions:

(i) The applicability of perturbation theory to calculate the density operator. There should at least be a region of small χ where perturbation theory works.

(ii) The heat bath should be big enough not to be influenced by the system. This is the usual assumption for a heat bath. Of course, this assumption eneds a small interaction between system and heat bath, too.

Let us discuss these two conditions for our model Hamiltonian (2.1). First of all, we see that Bose condensation appears even if the coupling constant χ is very small, as the typical parameter for this two-quanta process H_{aab} (1.6) is $N_0\chi$, which goes to infinity. The perturbation theory which yields the expansion of the state (2.6) in terms of χ certainly does not converge for $N_0\chi \to \infty$. Indeed, we have

$$|\beta\rangle \sim \exp\left[\sum_k \beta_k b_k^\dagger\right]|\ \rangle, \quad b_k|\ \rangle = 0 , \tag{5.1}$$

but β_k is

$$\beta_k = -\frac{MN_0\chi}{2\Omega_k} ,$$

which goes to infinity. We see that perturbation theory for the states and therefore for the density operator does not work either. On the other hand, the state for the heat bath has changed dramatically, which violates condition (ii).

We see two perspectives for the microscopic foundation of Fröhlich's theory. First, the Wu-Austin Hamiltonian could possibly be adjusted in order to provide the postulated rate equation. In its present form, this Hamiltonian cannot describe any realistic physical system (an infinite energy gap is unphysical). Second, it could be possible that there is no Hamiltonian at all that describes Fröhlich theory adequately. In any case we expect interesting discussions on these questions.

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